



Rocket Equation & Multistaging

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LECTURE FOR SATELLITE AND SPACE SYSTEMS MSC

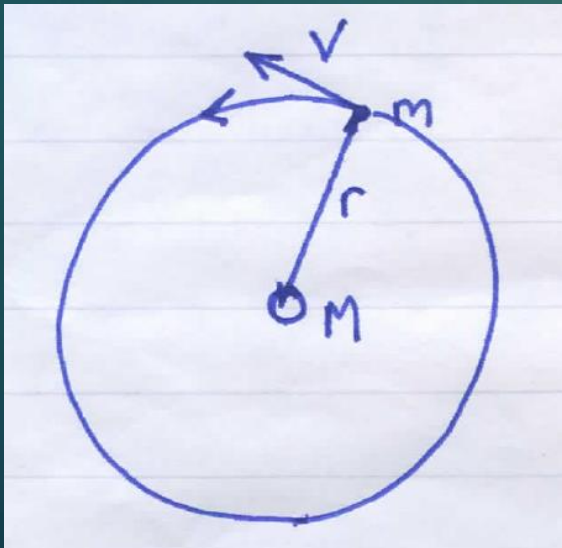
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25TH APRIL 2017

Orbital Mechanics & the Escape Velocity

- ▶ The motion of a space craft is that of a body with a certain momentum in a gravitational field
- ▶ The spacecraft moves under the combined effects of its momentum and the gravitational attraction towards the centre of the earth
- ▶ For a circular orbit $V = wr$ (1)



$$F = mrw^2 = \text{centripetal acceleration} \quad (2)$$

And this is balanced by the gravitational attraction

$$F = mMG / r^2 \quad (3)$$

$$w = \left(\frac{GM}{r^3}\right)^{1/2} \text{ or } V = \left(\frac{GM}{r}\right)^{1/2} \quad (4)$$

Orbital Velocity V

► $G = 6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2}$

Gravitational constant

► $M = 5.97 \times 10^{24} \text{ kg}$

Mass of the Earth

► $r_0 = 6.371 \times 10^6 \text{ m}$

► Hence $V = 7900 \text{ m/s}$

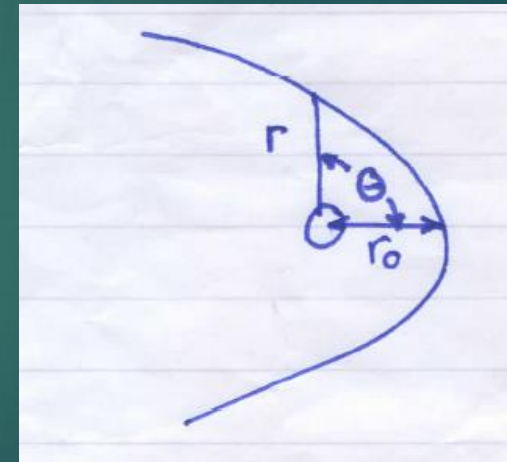
► For a non-circular (eccentric) orbit

► $\frac{1}{r} = \frac{GMm^2}{h^2} (1 + \varepsilon \cos\theta)$ (5)

► $h = mrV$ angular momentum (6)

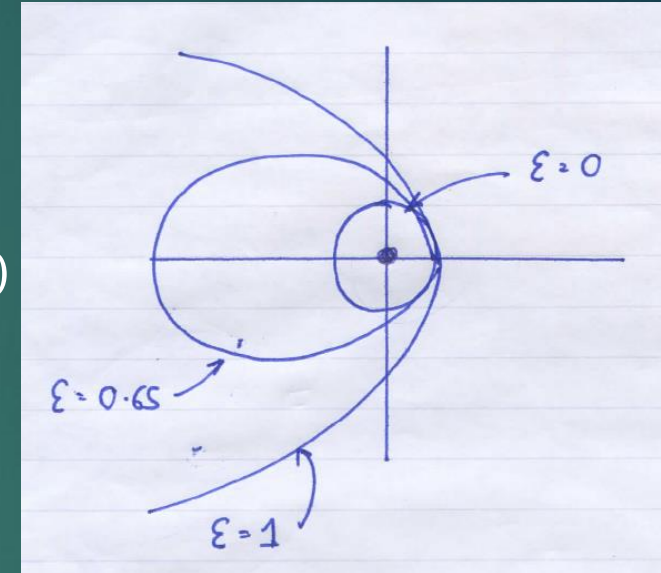
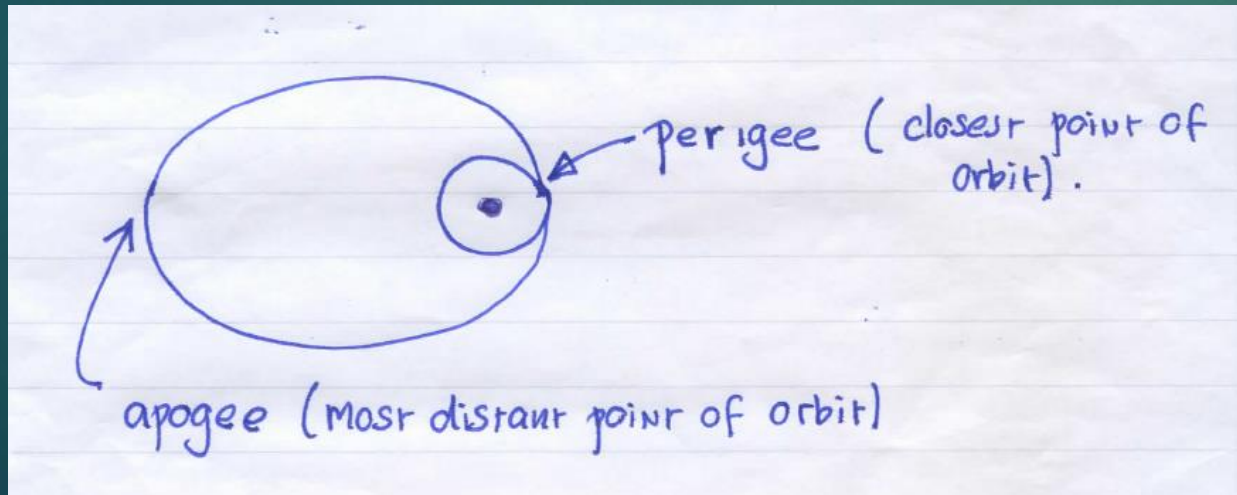
► $\varepsilon = \frac{h^2}{GMm^2r_0} - 1$ eccentricity (7)

► $V = \left[\frac{GM}{r} (1 + \varepsilon \cos\theta) \right]^{\frac{1}{2}}$ (8)



Orbit type

- ▶ For $\varepsilon = 0$ circular orbit
- ▶ For $\varepsilon = 1$ parabolic orbit
- ▶ For $\varepsilon > 1$ hyperbolic orbit (interplanetary fly-by)

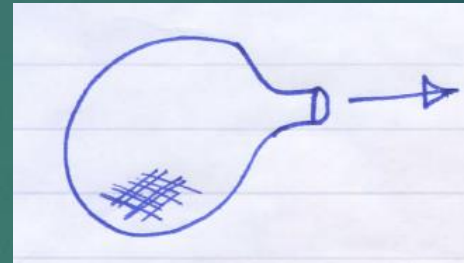


Escape Velocity

- ▶ For $\varepsilon = 1$ a parabolic orbit, the orbit ceases to be closed and the space vehicle will not return
- ▶ In this case at r_0 using equation (7)
- ▶ $1 = \frac{h^2}{GMm^2r_0} - 1$ but $h = mrv$ (9)
- ▶ $2 = \frac{m^2V_0^2r_0^2}{GMm^2r_0}$ therefore $V_0 = \left(\frac{2GM}{r_0}\right)^{\frac{1}{2}}$ (10)
- ▶ $G = 6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2}$ Gravitational constant
- ▶ $M = 5.975 \times 10^{24} \text{ kg}$ Mass of the Earth
- ▶ $r_0 = 6.371 \times 10^6 \text{ m}$ Mean earth radius
- ▶ $V_0 = \left(\frac{2GM}{r_0}\right)^{\frac{1}{2}}$ Escape velocity (11)
- ▶ $V_0 = 11,185 \text{ m/s}$ ($\sim 25,000 \text{ mph}$)
- ▶ $7,900 \text{ m/s}$ to achieve a circular orbit ($\sim 18,000 \text{ mph}$)

Newton's 3rd Law & the Rocket Equation

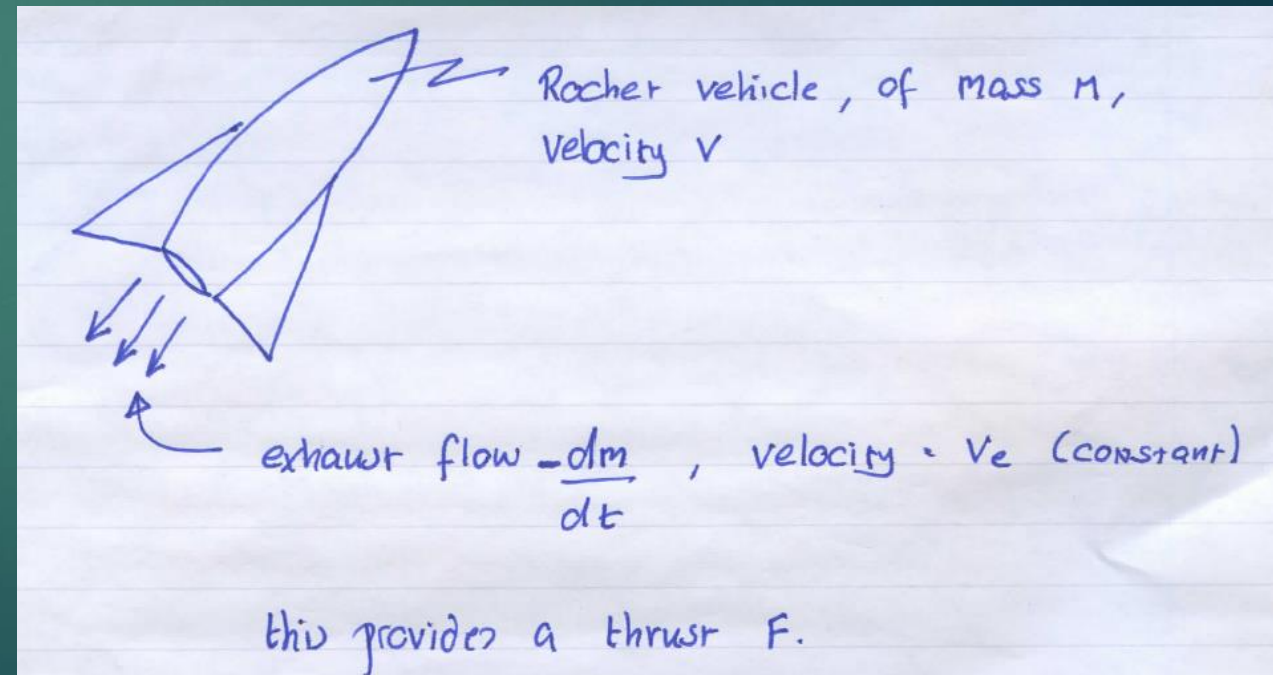
- ▶ N3: "to every action there is an equal and opposite reaction"
- ▶ A rocket is a device that propels itself by emitting a jet of matter.
- ▶ The momentum carried away results in a force acting to accelerate the rocket in a direction opposite to that of the jet
- ▶ Like a balloon expelling its gas and providing thrust



- ▶ A rocket is different to a gun because a bullet is given all its energy at the beginning of its flight. The energy of the bullet then decreases with time due to the losses against air friction.
- ▶ A cannon shell or a bullet is a projectile
- ▶ A rocket is a vehicle

Rocket Equation

- ▶ Thrust $F = -V_e \frac{dm}{dt}$ negative because the mass of the rocket decreases with time (14)
- ▶ The acceleration of the rocket under this force is given by Newton's 2nd Law
- ▶ $F = m \frac{dV}{dt}$ (15)
- ▶ Therefore $\frac{dV}{dt} = -\frac{1}{M} V_e \frac{dM}{dt}$ (16)
- ▶ $dV = -V_e \frac{dM}{M}$ (17)
- ▶ Integrate between limits of zero and V , for a change in mass M_0 to M gives the result



Rocket Equation

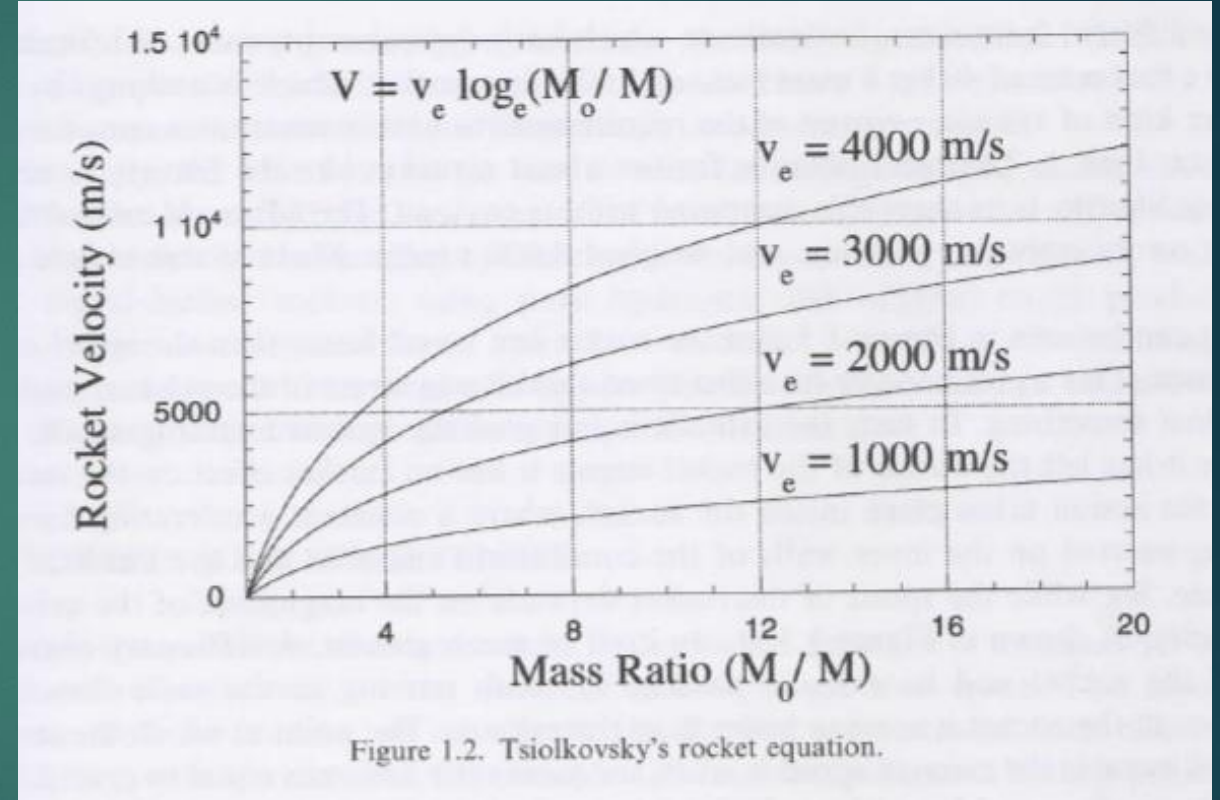
► $\int_0^V dV = -V_e \int_{M_0}^M \frac{dM}{M}$ (18)

► $V = -V_e \log_e \left(\frac{M}{M_0} \right)$ (19)

► $V = V_e \log_e \left(\frac{M_0}{M} \right)$ (20)

- This is Tsiolkovsk's Rocket Equation
- The rocket equation shows that the final speed depends upon only two numbers
 - The final mass ratio
 - The exhaust velocity

It does not depend on the thrust; size of engine; time of burn



Exit velocity depends on the fuel

- ▶ Gunpowder $V_e \approx 2000 \frac{m}{s}$
- ▶ Liquid fuel $V_e \approx 4500 \frac{m}{s}$
- ▶ Mass ratio = $\frac{\text{Vehicle mass} + \text{propellant mass}}{\text{Vehicle mass}} = \frac{M_0}{M}$ (21)
- ▶ $\frac{M_0}{M_V} = 20$ implies 95% of the initial mass is fuel (22)

A rocket can travel faster than its exhaust speed

- ▶ A rocket can travel faster than its exhaust speed V_e
- ▶ This appears to be counter intuitive if we think of the exhaust as pushing against something. But this is not the case
- ▶ All the action and reaction takes place inside the rocket where an accelerating force is being developed against the walls of the combustion chamber and the inside of the nozzle
- ▶ A rocket will exceed its exhaust speed when
- ▶ $\log_e \left(\frac{M_0}{M} \right) = 1$ (25)
- ▶ ie $\frac{M_0}{M} = e = 2.718$ (26)

Diminishing returns

- ▶ It is clear that increasing the mass ratio, that is: increasing the mass of fuel leads to diminishing returns
- ▶ For $V_e = 1000 \text{ m/s}$, $V \Rightarrow 3000 \text{ m/s}$
- ▶ A higher mass ratio will produce a higher velocity but only with a diminishing return
- ▶ To escape the earth's gravitational field a velocity of around 11 km/s is required.
- ▶ This can only be achieved with a high exhaust velocity and a large mass ratio

Gravity loss

- ▶ Our main result neglects the so called “gravity loss” that is the work done against gravity. If this were included
- ▶ $V = V_e \log_e \left(\frac{M_0}{M} \right) - gt$ (27)
- ▶ $V = V_e \log_e \left(\frac{M_0}{M} \right) - g \frac{M_0}{\dot{m}} \left(1 - \frac{M}{M_0} \right)$ (28)
- ▶ $g \frac{M_0}{\dot{m}} \left(1 - \frac{M}{M_0} \right)$ can account for 1200 m/s

Multi stage Rockets

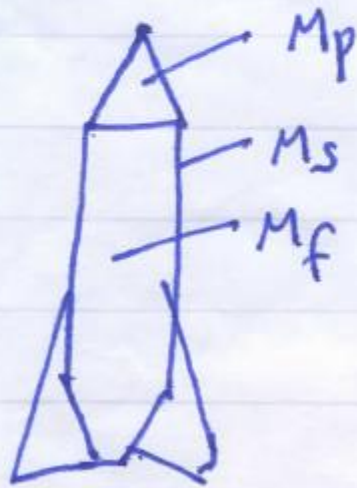
- ▶ As previously demonstrated a velocity of about 11 km/s is required to achieve escape from earth's gravity
- ▶ A velocity of 8 km/s is required to achieve a circular orbit
- ▶ For a single stage rocket, with modern fuel $V_e \sim 4$ km/s
- ▶ This implies a mass ratio of:
 - ▶ ~ 16 to achieve escape velocity $15/16^{\text{th}}$ fuel $\sim 94\%$
 - ▶ ~ 7.4 to achieve orbit $6.4/7.4$ fuel $\sim 86\%$
- ▶ Although the latter is currently possible, the former, ie escape velocity can only be achieved by multi stage rockets

Multi-stage rockets

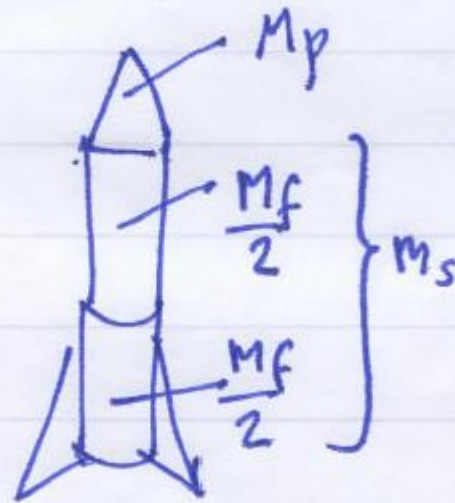
- ▶ For a single stage rocket
- ▶ $R_0 = \frac{M_0}{M} = \frac{M_S + M_F + M_P}{M_S + M_P}$
- ▶ M_F = fuel mass
- ▶ M_P = payload mass
- ▶ M_S = structural mass (depends on design – engines, pumps, fuel tanks, control systems)
- ▶ In general, we may expect the structural mass is kept to a minimum and is a constant proportion of the fuel mass for stages using the same fuel.

Multi-stage rockets

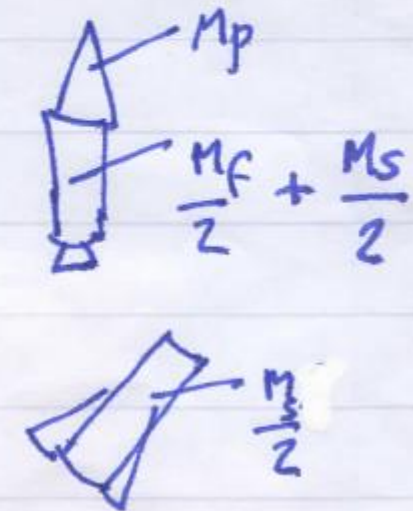
Single Stage
Rocket



two Stage
(at launch)



two stage
(at separation)



Two-stage rocket

- ▶ This rocket is divided into two stages
- ▶ The first rocket stage is ignited and burns until all its fuel is exhausted, this gives the whole stack a velocity defined by the rocket equation, with the mass ratio of:
- ▶ $R_1 = \frac{M_0}{M} = \frac{M_S + M_F + M_P}{M_S + \frac{M_F}{2} + M_P} \quad \frac{\text{Before}}{\text{After}}$
- ▶ The first stage burns out, is dropped off and the 2nd stage is ignited. It then gains additional velocity defined again by the rocket equation with mass ratio
- ▶ $R_2 = \frac{\frac{1}{2}M_S + \frac{1}{2}M_F + M_P}{\frac{1}{2}M_S + M_P} \quad \frac{\text{Before}}{\text{After}}$
- ▶ The second stage begins its burn with the payload, half the structural mass and half the fuel mass and ends with half the structural mass and the payload
- ▶ The final velocity is the sum of the two velocity increments

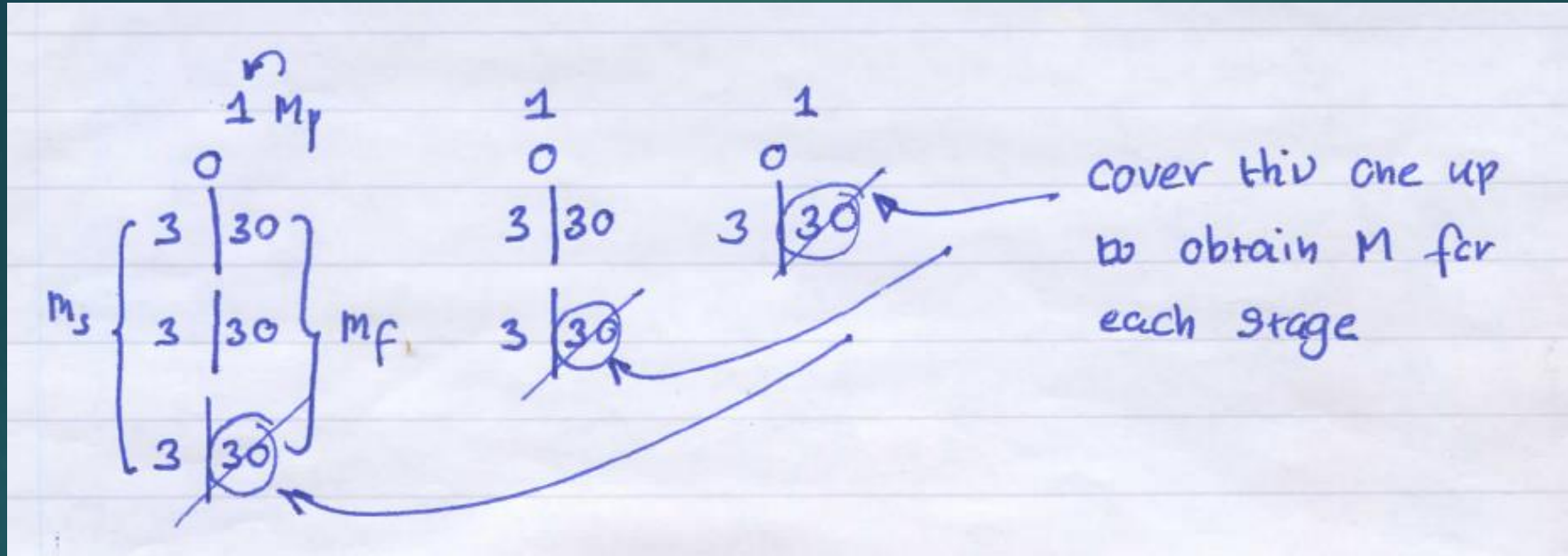
Single stage and two stage rocket compared

- ▶ So to compare the performance of single stage and two stage rockets we need to calculate:
- ▶ $V_0 = V_e \log_e(R_0)$
- ▶ $V = V_e \log_e(R_1) + V_e \log_e(R_2)$
- ▶ For example:
- ▶ Total mass of 100 tonnes; Payload of 1 tonne
- ▶ $V_e = 2.7 \times 10^3 \text{ m/s}$
- ▶ $M_s = 10\% \text{ of fuel mass}$
- ▶ Therefore $M_F = 90 \text{ tonnes}$; $M_s = 9 \text{ tonnes}$; $M_p = 1 \text{ tonne}$

Single stage and two stage rocket compared

- ▶ $V_0 = 2700 \log_e \left(\frac{9+90+1}{9+1} \right) = 6217 \text{ m/s}$ single stage final velocity
- ▶ Now divide the rocket into two smaller ones, each with half the fuel and the structural mass shared equally
- ▶ $V_1 = 2700 \log_e \left(\frac{9+90+1}{9+45+1} \right) = 1614 \text{ m/s}$
- ▶ $V_2 = 2700 \log_e \left(\frac{4.5+45+1}{4.5+1} \right) = 5986 \text{ m/s}$
- ▶ Total velocity increment = $V_1 + V_2 = 7600 \text{ m/s}$

Three stage rocket



- ▶ $R_1 = \frac{90+9+1}{60+9+1} = 1.4286; \quad V_1 = 2700 \log_e R_1 = 963 \text{ m/s}$
- ▶ $R_2 = \frac{60+6+1}{30+6+1} = 1.8108; \quad V_2 = 2700 \log_e R_2 = 1603 \text{ m/s}$
- ▶ $R_3 = \frac{30+3+1}{3+1} = 8.5; \quad V_3 = 2700 \log_e R_3 = 5778 \text{ m/s}$
- ▶ Total velocity increment = $V_1 + V_2 + V_3 = 963 + 1603 + 5778 = 8344 \text{ m/s}$

Multi stage rocket summary comparison

- ▶ Single stage Velocity increment = 6217 m/s
- ▶ Two stage Velocity increment = 7600 m/s
- ▶ Three stage Velocity increment = 8344 m/s

End